$x_1$, $x_2$, ...., $x_8$, as obtained from this system of equations.

Differentiating these equations, we have respectively,

$$\left(\frac{dr_1}{d\theta_1}\right)^2, \left(\frac{dr_2}{d\theta_2}\right)^2, \ldots, \left(\frac{dr_8}{d\theta_8}\right)^2 = \frac{b^2e^4\sin^2\theta_1\cos^2\theta_1}{(1-e^2\cos^2\theta_1)^3}, \frac{b^2e^4\sin^2\theta_2\cos^2\theta_2}{(1-e^2\cos^2\theta_2)^3},$$
$$\ldots, \frac{b^2e^4\sin^2\theta_8\cos^2\theta_8}{(1-e^2\cos^2\theta_8)^3} \ldots (2).$$

By means of the formula for the rectification of plane curves represented by polar co-ordinates, we have from (2)

$$\int_0^{l_1} ds_1 = b\int_0^{\theta_2} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_1]}}{(1-e^2\cos^2\theta_1)^{\frac{3}{2}}}\, d\theta_1;$$

$$\int_0^{l_2} ds_2 = b\int_0^{\theta_3} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_2]}}{(1-e^2\cos^2\theta_2)^{\frac{3}{2}}}\, d\theta_2;$$

$$\int_0^{l_3} ds_3 = b\int_0^{\theta_4} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_3]}}{(1-e^2\cos^2\theta_3)^{\frac{3}{2}}}\, d\theta_3;$$

$$\int_0^{l_4} ds_4 = b\int_0^{\pi} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_4]}}{(1-e^2\cos^2\theta_4)^{\frac{3}{2}}}\, d\theta_4;$$

$$\int_0^{l_5} ds_5 = b\int_0^{\theta_6} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_5]}}{(1-e^2\cos^2\theta_5)^{\frac{3}{2}}}\, d\theta_5;$$

$$\int_0^{l_6} ds_6 = b\int_{\pi}^{\theta_7} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_6]}}{(1-e^2\cos^2\theta_6)^{\frac{3}{2}}}\, d\theta_6;$$

$$\int_0^{l_7} ds_7 = b\int_{\pi}^{\theta_8} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_7]}}{(1-e^2\cos^2\theta_7)^{\frac{3}{2}}}\, d\theta_7;$$

$$\int_0^{l_8} ds_8 = b\int_{\pi}^{2\pi} \frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta_8]}}{(1-e^2\cos^2\theta_8)^{\frac{3}{2}}}\, d\theta_8.$$

The evaluation of the thirty-two integrals indicated in (1) is a labor sufficient to discourage even a mathematical Hercules.

**6. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

Find the average length of all the diameters that can be drawn in a given ellipse.

**Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Let $2r$ represent any diameter; then from the *central-polar* equation of the ellipse, $r^2 = \dfrac{b^2}{1-e^2\cos^2\theta}$, we have $2r = \dfrac{2b}{\sqrt{(1-e^2\cos^2\theta)}}$,

$$\frac{dr}{d\theta} = \frac{be^2\sin\theta\cos\theta}{(1-e^2\cos^2\theta)^{\frac{3}{2}}}; \text{ and } \frac{ds}{d\theta} = b\sqrt{\left(\frac{1-e^2(2-e^2)\cos^2\theta}{(1-e^2\cos^2\theta)^3}\right)}.$$

Since the number of diameters that can be drawn in an elliptic quadrant is proportional to the length of the elliptic arc bounding that quadrant, the required average length becomes

$$D=2b\int_0^{\frac{1}{2}\pi}\frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta]}}{(1-e^2\cos^2\theta)^2}\,d\theta\div\int_0^{\frac{1}{2}\pi}\frac{\sqrt{[1-e^2(2-e^2)\cos^2\theta]}}{(1-e^2\cos^2\theta)^{\frac{3}{2}}}\,d\theta.$$

Representing $e^2(2-e^2)$ by $p$ and expanding,

$$D=2b\int_0^{\frac{1}{2}\pi}\frac{(1-\frac{1}{2}p\cos^2\theta-\frac{1}{8}p^2\cos^4\theta-\frac{1}{16}p^3\cos^6\theta-\frac{5}{128}p^4\cos^8\theta-\text{etc.})}{(1-e^2\cos^2\theta)^2}d\theta$$

$$\div\int_0^{\frac{1}{2}\pi}\frac{(1-\frac{1}{2}p\cos^2\theta-\frac{1}{8}p^2\cos^4\theta-\frac{1}{16}p^3\cos^6\theta-\frac{5}{128}p^4\cos^8\theta-\text{etc.})}{(1-e^2\cos^2\theta)^{\frac{3}{2}}}\,d\theta$$

$$=2b\int_0^{\frac{1}{2}\pi}[1+\tfrac{1}{2}e^2(2+e^2)\cos^2\theta+\tfrac{1}{8}e^4(4+12e^2-e^4)\cos^4\theta+\tfrac{1}{16}e^6(-8+52e^2$$
$$-10e^4+e^6)\cos^6\theta+\tfrac{1}{128}e^8(-272+800e^2-264e^4+56e^6-5e^8)\cos^8\theta+\text{etc.}]d\theta$$

$$\div\int_0^{\frac{1}{2}\pi}[1+\tfrac{1}{2}e^2(1+e^2)\cos^2\theta+\tfrac{1}{8}e^4(-1+10e^2-e^4)\cos^4\theta+\tfrac{1}{16}e^6(-15+39e^2$$
$$-9e^4+e^6)\cos^6\theta+\tfrac{1}{128}e^8(-261+564e^2-222e^4+28e^6-5e^8)\cos^8\theta+\text{etc.}]d\theta$$

$$=2b\left(\frac{1+\frac{1}{4}e^2(2+e^2)+\frac{3}{64}e^4(4+12e^2-e^4)+\frac{5}{256}e^6(-8+52e^2-10e^4+e^6)}{1+\frac{1}{4}e^2(1+e^2)+\frac{3}{64}e^4(-1+10e^2-e^4)+\frac{5}{256}e^6(-15+39e^2-9e^4+e^6)}\right),$$

which is the required average length.

*Cor.*—Put $e=\frac{1}{2}$; then substitute and reduce, we obtain

$D=\dfrac{1218749}{1133057}$ of $2b=1.07563$ times $2b=1.076\times2b=\dfrac{269}{250}$(of the minor axis of the given ellipse).

This problem was also solved by Professors *SCHEFFER* and *ZERR*. Professor *MATZ* sent in three different solutions.

## PROBLEMS.

**13. Proposed by I. L. BEVERAGE, Monterey, Virginia**

Find the mean values of the roots of the quadratic $x^2-ax+b=0$, the roots being known to be real, but $b$ being unknown and positive.

**14. Proposed by CHARLES E. MYERS, Canton, Ohio.**

$\frac{1}{3}$ of all the mellons in a patch are not ripe, and $\frac{1}{4}$ of all the mellons in the same patch are rotten, the remainder being good. If a man enters the patch on a dark night and takes a mellon at random, what is the probability that he will get a good one?

**15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Todhunter proposes: "From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile; prove the chance of its falling within the field, is $C=2^{-1}-2\pi^{-1}(\sqrt{2}-1),=.236+.$" Is this result perfectly correct as to fact?